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A note on the geometry of connections in gauge theories of the electroweak interaction

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Abstract. It is shown that in gauge theories of the electroweak interaction, the symmetry-breaking process always defines a genuine connection on the electromagnetic sub-bundle.

1. Introduction

It is well known that the theory of fibre bundles provides a natural and useful framework for the formulation of classical field theory. This approach has significantly improved our understanding of a number of quite fundamental aspects of gauge theory, such as the relationship between the gauge transformations of the first and second kinds. More recently, the concept of symmetry breakdown has also been analysed in these terms (Madore 1977, Trautman 1979). If (PMG) is a principal fibre bundle over a space–time M, corresponding to a gauge group (structural group) G, and if H is a closed subgroup of G, then the breakdown of G to H may be interpreted as the reduction of (PMG) to a sub-bundle of the form (QMH). Such a reduction exists if and only if the associated bundle (EPMG, G/H) with standard fibre G/H admits a global cross-section $\theta: M \to E$ (Kobayashi and Nomizu 1963). This cross-section can also be interpreted as a global cross-section of a certain vector bundle. The pull-back (by means of local cross-sections of P) of the function on P to which this cross-section is equivalent, corresponds to the 'symmetry-breaking' scalar fields such as the Higgs fields (Trautman 1979). Thus, the geometrical approach allows us to understand the origin of the 'symmetry-breaking' fields.

Upon the breakdown of G to H, the gauge fields A^i_μ corresponding to G define two sets of new fields. Firstly, they define a set of massive fields with homogeneous transformation behaviour under the action of H; and secondly, they define the gauge fields corresponding to H. The geometrical meaning of this latter procedure has been formulated by Madore (1977) in terms of the theory of connections on principal bundles. Here we shall examine this question in detail in the case of gauge theories of the electroweak interaction. We shall prove that for all electroweak theories based on semisimple gauge groups or on groups of the form (semisimple) U(1), the gauge fields which survive symmetry breakdown do indeed define a genuine connection on the appropriate sub-bundle.

2. The splitting of the connection

Let (PMG) be a principal bundle over a space–time M, with structural group G. Let ω be a Lie algebra valued one-form on P, defining a connection Γ. The gauge potential and field tensor are then given as usual by the pull-backs $A=\sigma^*\omega$, $F=2\sigma^*D\omega$, where D denotes the covariant exterior derivative, and where σ is a local cross-section of P. Now let G break down to a closed subgroup H, so that a sub-bundle of the form (QMH) is defined. As remarked above, the connection in the sub-bundle is determined by that in (PMG): for example, in the model of Weinberg (1967) and Salam (1968), the photon field is constructed from the SU(2)×U(1) gauge fields. If $\{T_i\}$ is a basis for the Lie algebra of SU(2)×U(1) (with T_4 as the hypercharge generator) then the connection form on the SU(2)×U(1) bundle may be written as

$$\begin{aligned}\omega&=\omega^1T_1+\omega^2T_2+\omega^3T_3+\omega^4T_4\\&=\omega^1T_1+\omega^2T_2+\tfrac{1}{2}(\omega^3+\omega^4)(T_3+T_4)+\tfrac{1}{2}(\omega^3-\omega^4)(T_3-T_4).\end{aligned}\tag{1}$$

Regarding $\frac{1}{2}(T_3+T_4)$ as the generator of the electromagnetic subgroup, we see that the coefficient of the 'charge component' of ω is just $\omega^3+\omega^4$. But this form, when pulled back to space–time, is just the electromagnetic field (potential) $A^3_\mu+A^4_\mu$. That is, the connection in the electromagnetic sub-bundle is just the charge component of the restriction of ω to that sub-bundle.

Returning to the previous case, Madore (1977) has pointed out that this observation is quite general. If (PMG) and (QMH) are as above, we can regard the Lie algebra $\tilde{\mathrm{H}}$ of H as a subalgebra of the Lie algebra $\tilde{\mathrm{G}}$ of G. Splitting $\tilde{\mathrm{G}}$ into a direct sum $\tilde{\mathrm{G}}=\tilde{\mathrm{H}}\oplus N$, we can also split ω in the same way. The gauge fields corresponding to H are obtained as the pull-backs, by means of local cross-sections of Q, of the $\tilde{\mathrm{H}}$-component of the restriction of ω to Q. Denoting this restriction by ω_Q, we write

$$\omega_Q=\bar{\omega}+\lambda,\tag{2}$$

where $\bar{\omega}$ and λ are the $\tilde{\mathrm{H}}$ and N components of ω_Q. The question now is whether $\bar{\omega}$ is a connection form on the sub-bundle (QMH). The conditions for this will be discussed in the next section.

3. Conditions on G and H. The electroweak case

The question of whether $\bar{\omega}$, defined as above, does in fact define a connection on (QMH), depends only on the choice of G and H. It is easily shown (Kobayashi and Nomizu 1963) that if $g\to\mathrm{ad}(g)_*$ denotes the adjoint representation of G on $\tilde{\mathrm{G}}$, then the condition that $\bar{\omega}$ be a connection is that $\mathrm{ad(H)}_*N=N$. (The same proof shows that λ is a tensorial one-form of type (ad, N) under this condition. This means that the components of the pull-back of λ have a homogeneous transformation law under the action of H. These are clearly the massive 'gauge' fields which result from symmetry breaking.) Our aim here is to show that this condition is always met in the case of electroweak theories.

By definition of the Lie derivative, and since the flow of any element $A\in\tilde{\mathrm{G}}$ is of the form $\{R_{\mathrm{a}_t}\}$ (where a_t is the one-parameter subgroup generated by A), we have

$$[A,B]=\lim_{t\to0}(B-\mathrm{ad}(\mathrm{a}_t^{-1})_*B)/t,\tag{3}$$

where $B \in \tilde{G}$. From this it is clear that the condition $\mathrm{ad}(H)_* N = N$ is equivalent to the condition

$$[\tilde{H}, N] \subseteq N, \tag{4}$$

provided that H is connected, which we shall assume to be the case (so that any element of H can be expressed as a product of factors drawn from the one-parameter subgroups generated by the elements of $\tilde{H}$).

Now let us consider the specific case of electroweak theories, in which G is semisimple or of the form (semisimple) × U(1), and H is a U(1) subgroup generated by some specific linear combination of elements drawn from the Cartan subalgebra of G. Let us consider first the case in which G is semisimple. If $\tilde{G}_1$ is the Cartan subalgebra, then set $\tilde{G} = \tilde{G}_1 \oplus \tilde{G}_2$. Now it is well known (see, for example, Goto and Grosshans 1978) that $\tilde{G}$ has a 'root space decomposition' of the form

$$\tilde{G} = L(0) \oplus L(\alpha_1) \oplus L(\alpha_2) \oplus \ldots \tag{5}$$

where the $L(\alpha_i)$ (with $\alpha_0 = 0$) satisfy

$$[L(\alpha_i), L(\alpha_j)] \subseteq L(\alpha_i + \alpha_j), \tag{6}$$

and where L(0) is the Cartan subalgebra. Hence it is clear that $[\tilde{G}_1, \tilde{G}_2] \subseteq \tilde{G}_2$. Now we also have $\tilde{G} = \tilde{H} \oplus N$, where $\tilde{H}$ is a one-dimensional space spanned by the charge operator. Setting $\tilde{G}_1 = \tilde{H} \oplus R$, we have $N = R \oplus \tilde{G}_2$. Since R is a subspace of the Cartan subalgebra, which is Abelian, we have $[\tilde{H}, N] = [\tilde{H}, \tilde{G}_2]$. Since $\tilde{H} \subseteq \tilde{G}_1$, this means that $[\tilde{H}, N] \subseteq [\tilde{G}_1, \tilde{G}_2]$, and so

$$[\tilde{H}, N] \subseteq \tilde{G}_2 \subseteq N. \tag{7}$$

Thus the condition (4) is satisfied in this case. The case in which the Lie algebra is of the form (semisimple) × U(1) may be treated similarly, since the hypercharge operator commutes with all the other generators. The extension to groups such as SU(2) × SU(2) × U(1) is straightforward.

4. Conclusion

If (PMG) is a principal bundle with connection form ω, and if (QMH) is a reduced sub-bundle, then it is natural to attempt to define a connection on (QMH) by taking the connection form $\bar{\omega}$ on Q to be the $\tilde{H}$ component of the restriction of ω to Q. This, indeed, is the way in which the gauge fields corresponding to the gauge symmetry H arise from those corresponding to G, subsequent to the breakdown of G to H. In fact, however, $\bar{\omega}$ defined in this way does not invariably define a connection on the sub-bundle: in general, it will only do so if the condition $\mathrm{ad}(H) * N = N$ is met, where $\tilde{G} = \tilde{H} \oplus N$. This condition is also needed in order that the massive fields, which correspond to the N-component of the restriction of ω to Q, should have the correct transformation behaviour under the action of H. We have found that in the cases in which G is semisimple or of the form (semisimple) × U(1), and in which H is a U(1) generated by a linear combination of elements drawn from the Cartan subalgebra of the Lie algebra of G, this condition is always satisfied. Thus, for gauge theories of the electroweak interaction, the symmetry-breaking process always yields a true connection on the electromagnetic sub-bundle, that is, a photon field with the correct gauge transformation law. The situation as regards grand unified theories, in which the residual symmetry is typically $SU(3) \times U(1)$, remains to be examined.

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